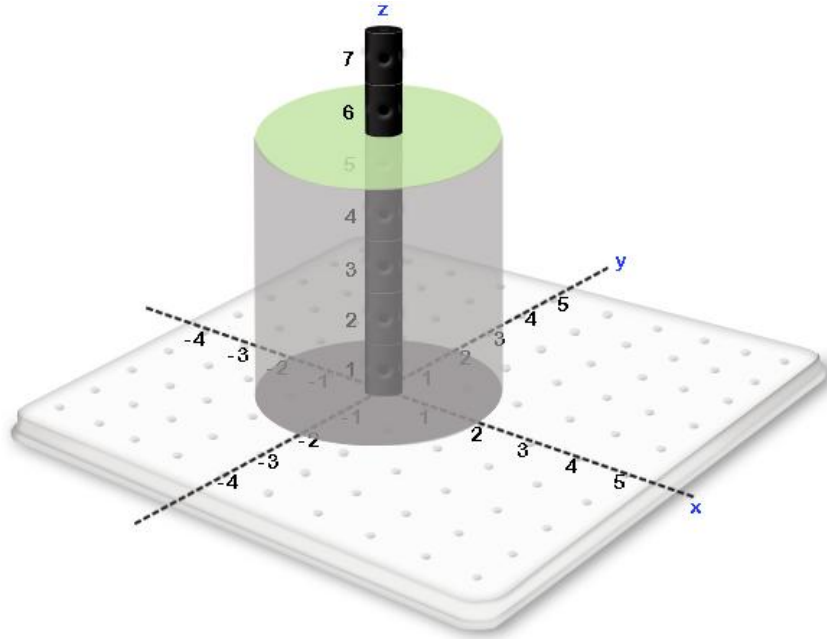


CHAPTER 9:

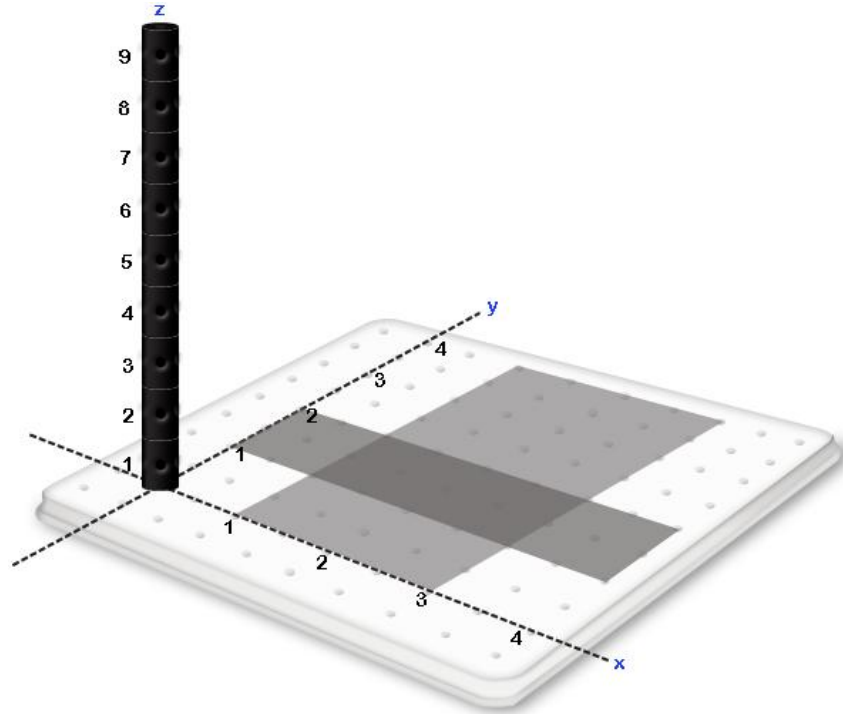
POLAR COORDINATES

9.1 INTRODUCTION

In Section 8.1, we saw an example of the volume above the region $x^2 + y^2 \leq 4$ and below the surface $f(x, y) = 5$.



If we wish to express this volume as a double integral in rectangular coordinates, we obtain $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) dy dx$ which is a very complex expression for a simple cylinder. If we consider a the set of points satisfying $a \leq x \leq b$ and $c \leq y \leq d$ where a, b, c and d are all constants, this region will be a rectangle. For Example, the set of points satisfying $1 \leq x \leq 3$ and $1 \leq y \leq 2$ is the rectangular region shown below.



Hence, identifying points with rectangular coordinates is very convenient for expressing rectangles but is very inconvenient for expressing circular regions. Hence, when working with circular regions, we would like to identify points with a new set of coordinates (r, θ) such that appropriate constants a, b, c and d will produce a circular region for the set of points satisfying $a \leq r \leq b$ and $c \leq \theta \leq d$. Polar coordinates satisfy this need and will be the subject of this chapter.

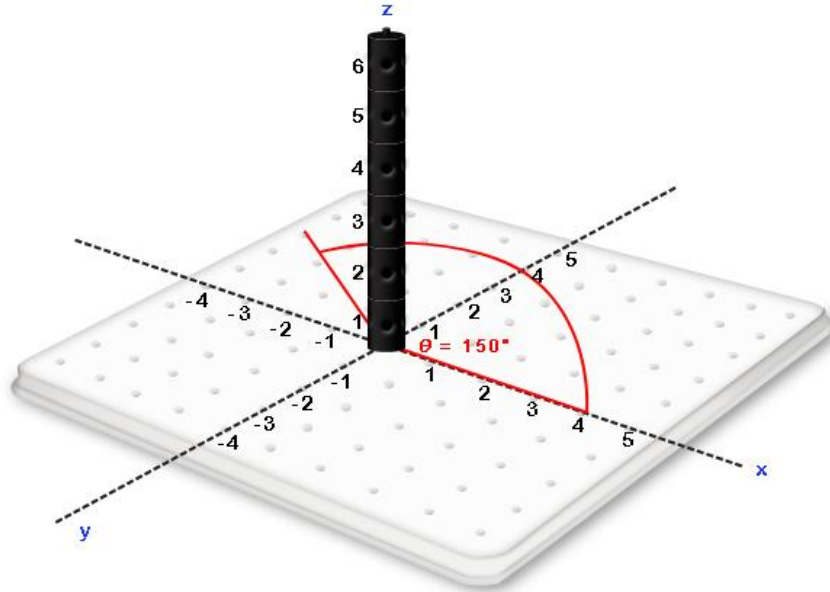
9.2 DEFINITION OF POLAR COORDINATES

A location on the xy plane can be defined with (r, θ) where

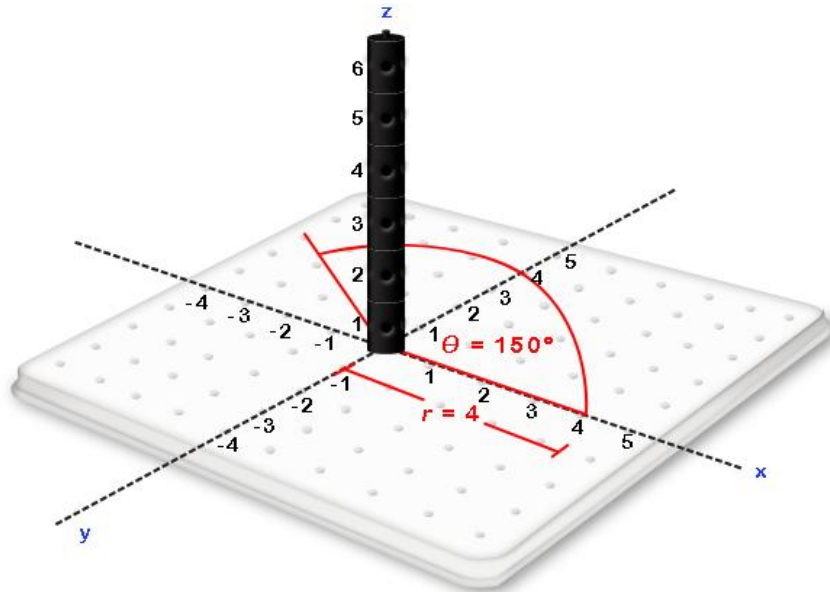
- θ is the angle between the vector $\langle 1, 0 \rangle$ placed at the origin and the vector that starts at $(0, 0)$ and ends at the given location and
- r is the distance from the location to the point $(0, 0)$

Example Exercise 9.2.1: Find the point $(r, \theta) = (150^\circ, 4)$.

Solution: Starting with $\theta = 150^\circ$ we can obtain all the points consistent with $\theta = 150^\circ$ on the xy plane.



Of those points where $\theta = 150^\circ$ on the xy plane, we can find the point that is 4 units away from the origin. I.e., the point consistent with $r = 4$.

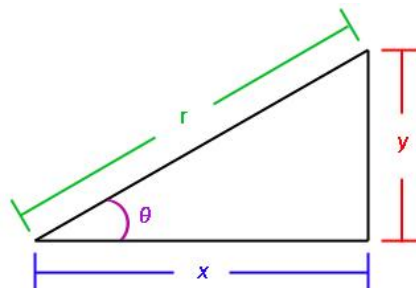


This is the point $(r, \theta) = (150^\circ, 4)$.

9.3 CONVERTING BETWEEN RECTANGULAR AND POLAR COORDINATES

As we saw in section 9.1, circular regions expressed in rectangular coordinates can be unnecessarily complicated. Hence, there are times where it will be advantageous to convert a

problem that is expressed in rectangular coordinates to polar coordinates. The following diagram shows a location in both rectangular and polar coordinates.



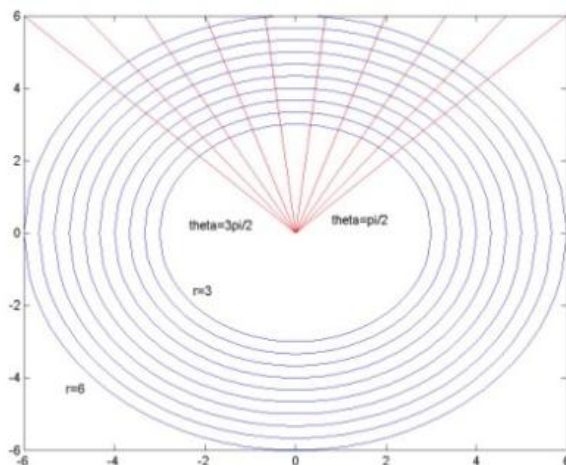
From this diagram, we can obtain the following relations that are useful to pass between rectangular and polar coordinates:

- $x = r\cos(\theta)$,
- $y = r\sin(\theta)$,
- $x^2 + y^2 = r^2$ or $r = \sqrt{x^2 + y^2}$ and
- $\tan(\theta) = \frac{y}{x}$ or $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ when (x, y) is in the first or fourth quadrant and $\theta = \tan^{-1}\left(\frac{y}{x}\right) + \pi$ when the point (x, y) is in the second or third quadrant.

9.4 POLAR REGIONS

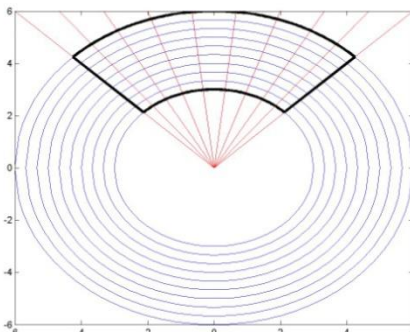
Example Exercise 9.4.1: Find the polar region associated with $3 \leq r \leq 6$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$.

Solution:



In the above graph, the set of points where $r = 3$, $r = 6$ and r equal to various values between 3 and 6 are shown in blue. The set of points where $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$ and θ equal to various values

between $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ are shown in red. The set of points where both $3 \leq r \leq 6$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ will be the intersection of the blue and red regions as is outlined in black in the following diagram.

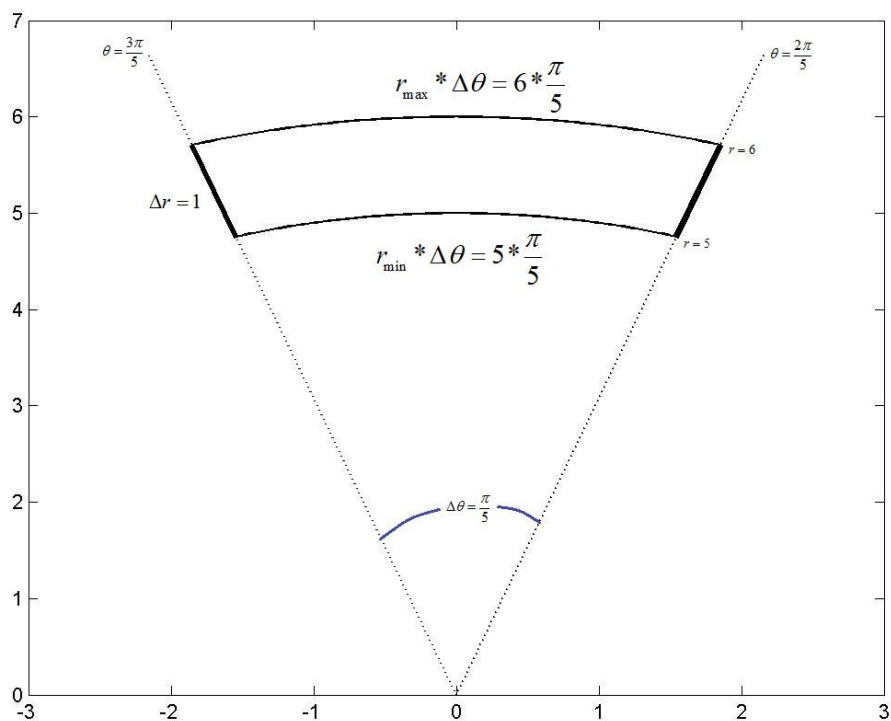


Hence polar coordinates allow us to define circular regions by letting the variables r and θ go from one constant to another constant. For example, if we wish to define the region associated with a circle of radius 4 centered at $(0, 0)$, we can select the set of points satisfying $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$.

9.5 APPROXIMATE THE AREA OF A POLAR RECTANGLE

The region associated with the set of points where r and θ go from one constant to another constant is often referred to as a polar rectangle.

If we consider the polar rectangle $5 \leq r \leq 6$ and $\frac{2\pi}{5} \leq \theta \leq \frac{3\pi}{5}$ shown in the following diagram, we can visualize that as $\Delta\theta \rightarrow 0$, the left and right sides of the polar rectangle below will approximate two parallel lines with the same length. We can also see that as $\Delta r \rightarrow 0$, the top and bottom sides will approach having the same length. Hence, as $\Delta r \rightarrow 0$ and $\Delta\theta \rightarrow 0$, a polar rectangle will come to resemble a rectangle with area length times width. As a result, it is not uncommon to approximate the area of a polar rectangle with length times width.



In the polar rectangle above, the top and bottom sides are two arcs with angle $\Delta\theta = \frac{\pi}{5}$. The bottom arc has radius $r = 5$ and the top arc has radius $r = 6$. Remembering that the length of an arc is given by the angle of the arc measured in radians multiplied by the radius of the arc, we find that the bottom arc has length $5 * \frac{\pi}{5} = \pi$ and the top arc has length $6 * \frac{\pi}{5}$. Both the left and right sides both have $\Delta r = 1$. Hence if we use smallest r associated with the region, the approximate area of the polar rectangle will be $length * width = \pi * 1$. If we use the largest r associated with the region, the approximate area of the polar rectangle will be $length * width = (6 * \frac{\pi}{5}) * 1$.

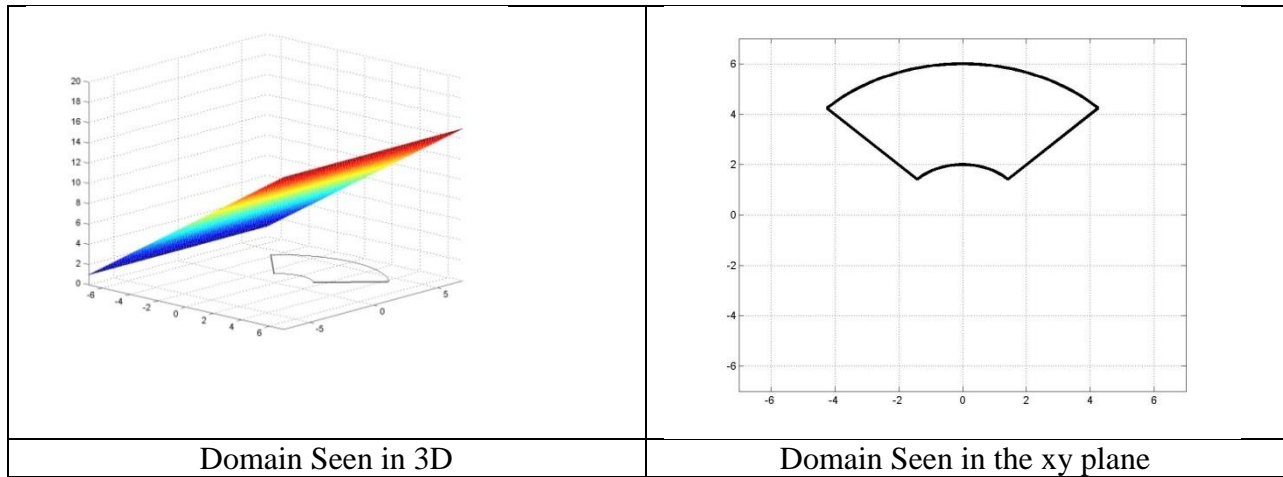
In general, we can approximate the area of the polar rectangle $r_1 \leq r \leq r_2$ and $\theta_1 \leq \theta \leq \theta_2$ with $area = R\Delta r\Delta\theta$ with $R = r_1$, $R = r_2$ or R being any r associated with the region that we have selected. As $\Delta r \rightarrow 0$ and $\Delta\theta \rightarrow 0$, this approximation becomes precise.

9.6 USING RIEMANN SUMS TO APPROXIMATE VOLUMES WITH POLAR COORDINATES.

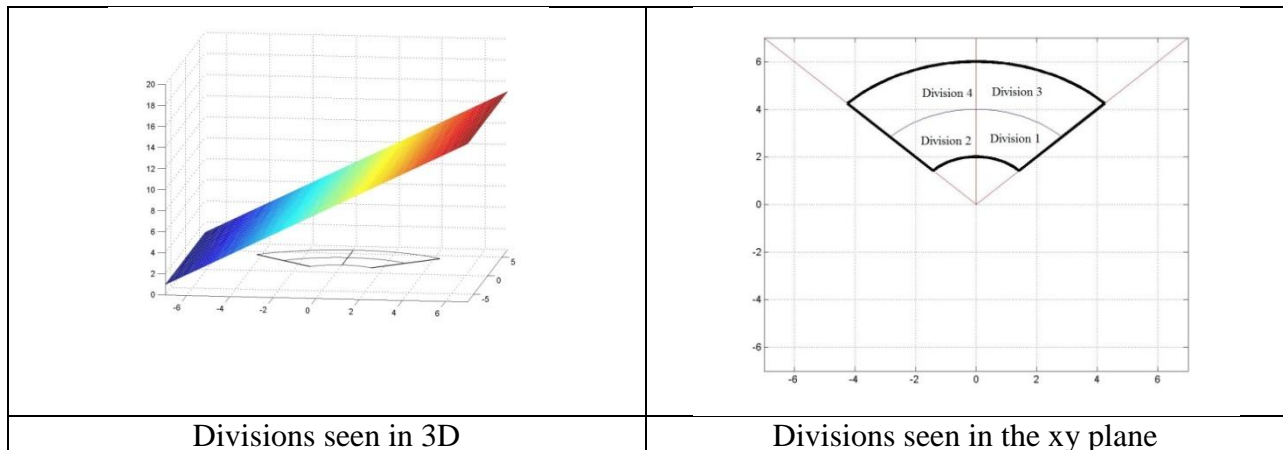
Example Exercise 9.6.1: Use Riemann Sums with two divisions in r and θ to approximate the volume over the region $2 \leq r \leq 6$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ in the xy plane and below the plane $z = x + 8 = r\cos(\theta) + 8$. Use the smallest value of each variable to represent a given division.

Solution:

Step 1: Identify the domain associated with the volume to be obtained.

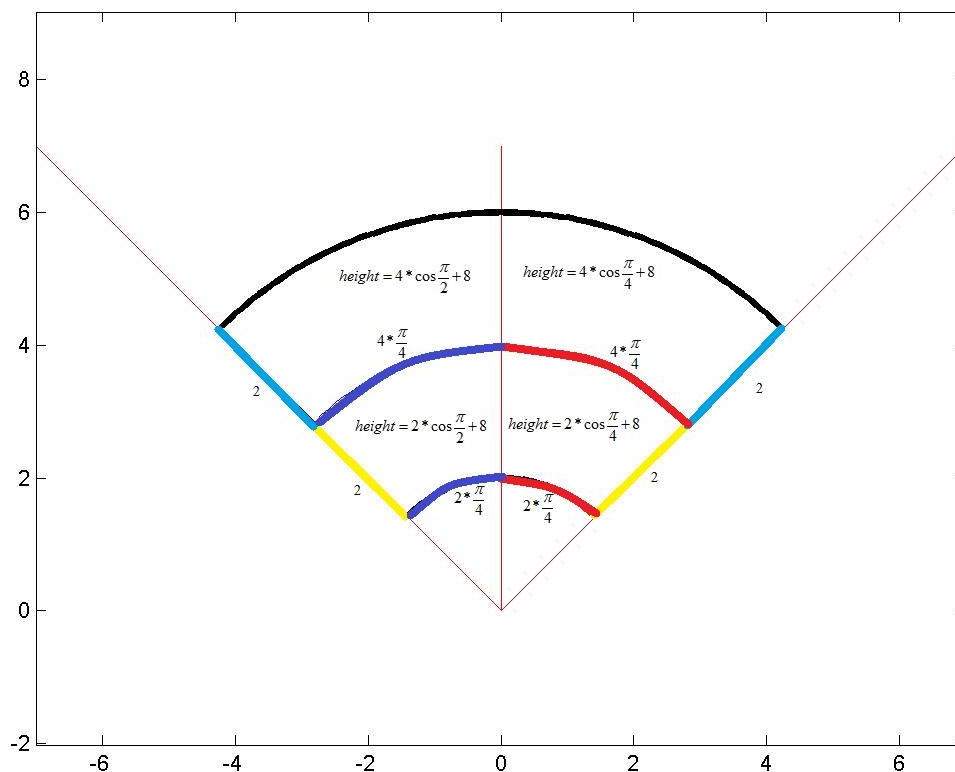


Step 2: Divide r and θ into two parts.



Step 3: Determine the volume of each division that will result when using the minimum value of each variable to obtain the length width and height:

For all four divisions, $\Delta\theta = \frac{\pi}{4}$ and $\Delta r = 2$. In divisions 1 and 2, r goes from 2 to 4 and in divisions 3 and 4 r goes from 4 to 6. In divisions 1 and 3, θ goes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ and in divisions 2 and 4, θ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$. In section 9.4 we determined that if Δr and $\Delta\theta$ are small, we can approximate the area of a polar rectangle by treating it as a rectangle with width Δr and length $r\Delta\theta$. The height of the surface above each division is given by $z = x + 8 = r\cos(\theta) + 8$. Hence for each division we can approximate the length width and height using the smallest value for r and θ found in each region as shown in the following diagram.



The length, width, height and volume of each division are summarized in the following table:

| Division | Length | Width | Height | Volume |
|----------|-------------------------------------|-------|--|----------------------|
| 1 | $2 * \frac{\pi}{4} = \frac{\pi}{2}$ | 2 | $2 \cos\left(\frac{\pi}{4}\right) + 8 = \sqrt{2} + 8$ | $\pi(\sqrt{2} + 8)$ |
| 2 | $2 * \frac{\pi}{4} = \frac{\pi}{2}$ | 2 | $2 \cos\left(\frac{\pi}{2}\right) + 8 = 8$ | 8π |
| 3 | $4 * \frac{\pi}{4} = \pi$ | 2 | $4 \cos\left(\frac{\pi}{4}\right) + 8 = 2\sqrt{2} + 8$ | $2\pi(\sqrt{2} + 8)$ |
| 4 | $4 * \frac{\pi}{4} = \pi$ | 2 | $4 \cos\left(\frac{\pi}{2}\right) + 8 = 8$ | 16π |

Step 4. Add the volumes of the four divisions to approximate the total volume of the solid:

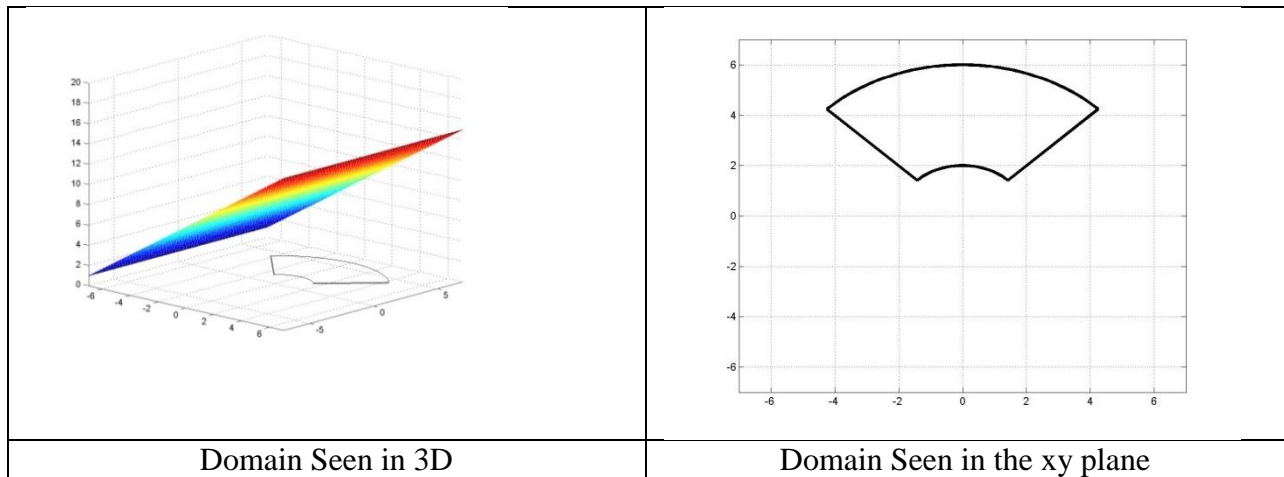
$$\text{Volume} \approx \pi(\sqrt{2} + 8) + 8\pi + 2\pi(\sqrt{2} + 8) + 16\pi$$

9.7 USING RIEMANN SUMS AND THE FUNDAMENTAL THEOREM FIND PRECISE VOLUMES

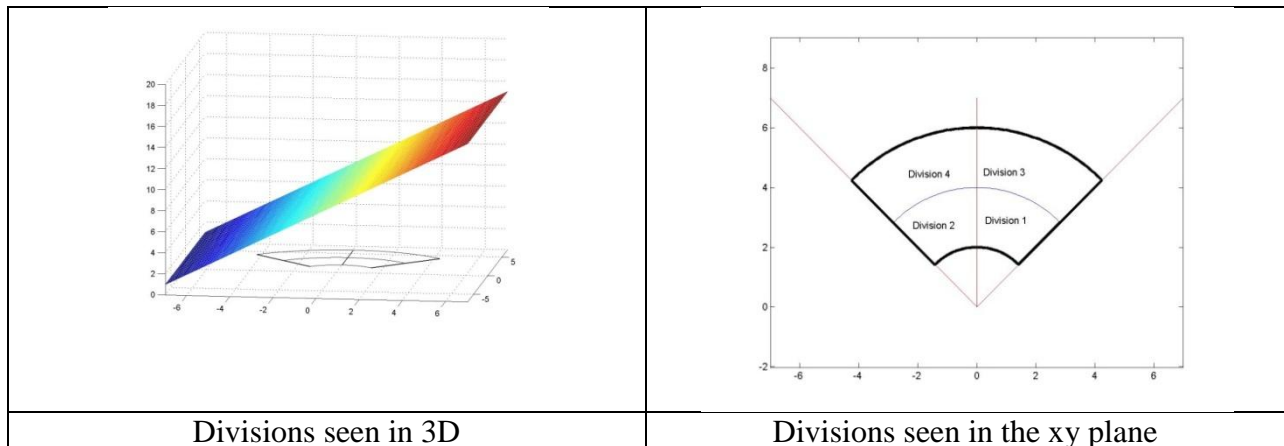
Example Exercise 9.7.1: Use Riemann Sums with two divisions in r and θ to approximate the volume over the region $2 \leq r \leq 6$ and $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ in the xy plane and below the plane $z = x + 8 = r \cos(\theta) + 8$. Use the smallest value of each variable to represent a given division.

Solution:

Step 1: Identify the domain associated with the volume to be obtained.



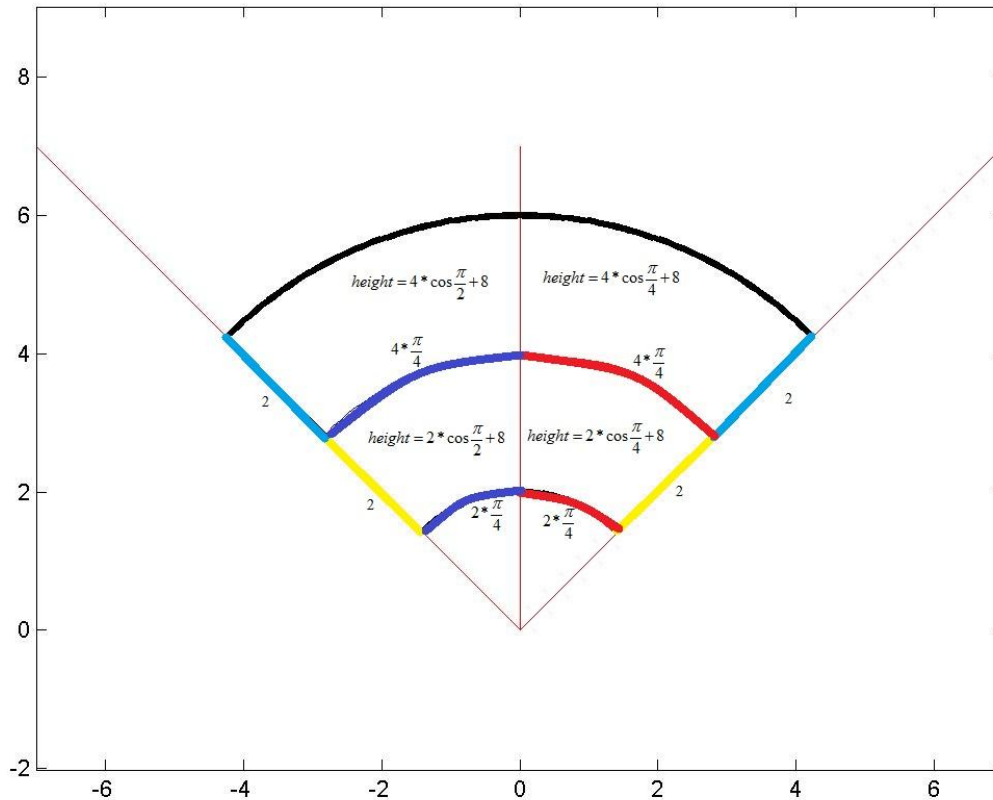
Step 2: Divide r and θ into two parts identifying $r_1, r_2, \theta_1, \theta_2, \Delta r$ and $\Delta \theta$:



In divisions 1 and 2, r goes from 2 to 4 and in divisions 3 and 4 r goes from 4 to 6. In divisions 1 and 3, θ goes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ and in divisions 2 and 4, θ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$. Hence, $r_1 = 2, r_2 = 4, \theta_1 = \frac{\pi}{4}, \theta_2 = \frac{\pi}{2}, \Delta r = 2$ and $\Delta \theta = \frac{\pi}{4}$

Step 3: Determine the numeric volume of each division that will result when using the minimum value of each variable to obtain the length width and height:

For all four divisions, $\Delta\theta = \frac{\pi}{4}$ and $\Delta r = 2$. In divisions 1 and 2, r goes from 2 to 4 and in divisions 3 and 4 r goes from 4 to 6. In divisions 1 and 3, θ goes from $\frac{\pi}{4}$ to $\frac{\pi}{2}$ and in divisions 2 and 4, θ goes from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$. In section 9.4 we determined that if Δr and $\Delta\theta$ are small, we can approximate the area of a polar rectangle by treating it as a rectangle with width Δr and length $r\Delta\theta$. The height of the surface above each division is given by $z = x + 8 = r\cos(\theta) + 8$. Hence for each division we can approximate the length width and height using the smallest value for r and θ found in each region as shown in the following diagram.



The length, width, height and volume of each division are summarized in the following table:

| Division | Length | Width | Height | Volume |
|----------|-------------------------------------|-------|--------------------------------------|---------------------|
| 1 | $2 * \frac{\pi}{4} = \frac{\pi}{2}$ | 2 | $2 * \cos(\pi/4) + 8 = \sqrt{2} + 8$ | $\pi(\sqrt{2} + 8)$ |

| | | | | |
|---|-------------------------------------|---|---------------------------------------|----------------------|
| 2 | $2 * \frac{\pi}{4} = \frac{\pi}{2}$ | 2 | $2 * \cos(\pi/2) + 8 = 8$ | 8π |
| 3 | $4 * \frac{\pi}{4} = \pi$ | 2 | $4 * \cos(\pi/4) + 8 = 2\sqrt{2} + 8$ | $2\pi(\sqrt{2} + 8)$ |
| 4 | $4 * \frac{\pi}{4} = \pi$ | 2 | $4 * \cos(\pi/2) + 8 = 8$ | 16π |

Step 4: Add the volumes of the four divisions to approximate the total volume of the solid:

$$Volume \approx \pi(\sqrt{2} + 8) + 8\pi + 2\pi(\sqrt{2} + 8) + 16\pi$$

Step 5: Find the length, width, height and volume of each division using $r_1, r_2, \theta_1, \theta_2, \Delta r$ and $\Delta\theta$ whenever it is appropriate.

| Division | Length | Width | Height | Volume |
|----------|----------------------|------------|-------------------------------|---|
| 1 | $r_1 * \Delta\theta$ | Δr | $r_1 * \cos(\theta_{11}) + 8$ | $(r_1 * \cos(\theta_{11}) + 8) r_1 \Delta r \Delta\theta$ |
| 2 | $r_1 * \Delta\theta$ | Δr | $r_1 * \cos(\theta_{12}) + 8$ | $(r_1 * \cos(\theta_{12}) + 8) r_1 \Delta r \Delta\theta$ |
| 3 | $r_2 * \Delta\theta$ | Δr | $r_2 * \cos(\theta_{21}) + 8$ | $(r_2 * \cos(\theta_{21}) + 8) r_2 \Delta r \Delta\theta$ |
| 4 | $r_2 * \Delta\theta$ | Δr | $r_2 * \cos(\theta_{22}) + 8$ | $(r_2 * \cos(\theta_{22}) + 8) r_2 \Delta r \Delta\theta$ |

Step 6: Add the volumes of the four divisions in table to approximate the total volume of the solid using $r_1, r_2, \theta_1, \theta_2, \Delta r$ and $\Delta\theta$:

$$Volume \approx (r_1 * \cos(\theta_1) + 8)r_1 \Delta r \Delta\theta + (r_1 * \cos(\theta_2) + 8) r_1 \Delta r \Delta\theta + (r_2 * \cos(\theta_1) + 8)r_2 \Delta r \Delta\theta + (r_2 * \cos(\theta_2) + 8) r_2 \Delta r \Delta\theta$$

Step 7: Express the approximate volume in Step 6 as a double summation:

$$\begin{aligned} Volume &\approx \{(r_1 \cos(\theta_{11}) + 8) r_1 \Delta r \Delta\theta + (r_1 \cos(\theta_{12}) + 8) r_1 \Delta r \Delta\theta\} + \\ &\quad \{(r_2 \cos(\theta_{21}) + 8) r_2 \Delta r \Delta\theta + (r_2 \cos(\theta_{22}) + 8) r_2 \Delta r \Delta\theta\} \\ &= \sum_{j=1}^2 (r_1 \cos(\theta_j) + 8) r_1 \Delta r \Delta\theta + \sum_{j=1}^2 (r_2 \cos(\theta_j) + 8) r_2 \Delta r \Delta\theta \\ &= \sum_{i=1}^2 \sum_{j=1}^2 (r_i \cos(\theta_j) + 8) r_i \Delta\theta \Delta r \end{aligned}$$

Step 8: From section 9.5, we know that as $\Delta r \rightarrow 0$ and $\Delta\theta \rightarrow 0$, this approximation becomes precise so apply the fundamental theorem to the double sum in step 7 to find the precise volume of the solid:

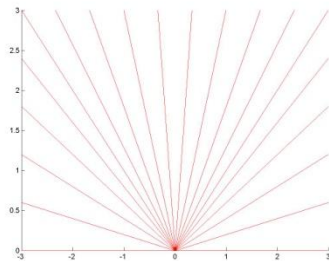
$$Volume = \lim_{\Delta r \rightarrow 0} \lim_{\Delta\theta \rightarrow 0} \sum_{i=1}^2 \sum_{j=1}^2 (r_i \cos(\theta_j) + 8) r_i \Delta\theta \Delta r$$

$$Volume = \int_2^6 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (r \cos(\theta) + 8) r dr d\theta$$

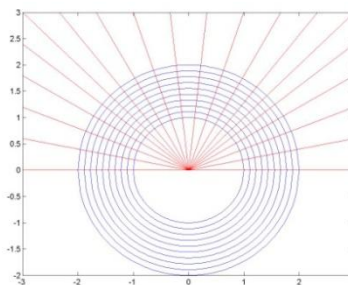
9.8 VOLUMES ASSOCIATED WITH INTEGRALS IN POLAR COORDINATES

Example Exercise 9.8.1: Find the volume associated with $\int_0^{\pi} \int_1^2 (r^2) r dr d\theta$

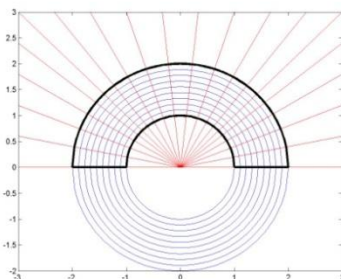
Solution: From section 9.7 we should remember that $\iint r dr d\theta$ corresponds to $\sum \sum r_i \Delta\theta \Delta r$ which represents a region in the xy plane and $z = r^2$ represents the height of the surface over this region. Working from the outside inward, the first datum from the integral is $\int_0^{\pi} \dots d\theta$ indicating that our volume will reside between $\theta = 0$ and $\theta = \pi$.



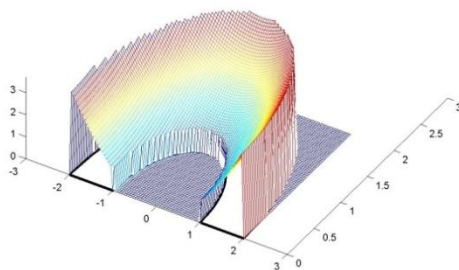
The second datum from the integral is $\int_1^2 dr$ indicating that for every value of θ between $\theta = 0$ and $\theta = \pi$, we will accept values of r that between $r = 1$ and $r = 2$.



The combination of the two data defined thus far give the following region in the xy plane.



The volume represented by will be the volume above this region and below $z = r^2$. Using the relations shown in 9.4, we can see that the surface above the region is $z = r^2 = x^2 + y^2$ which is a paraboloid that we have seen many times is and that we should be familiar with. The surface over the region is shown in the following diagram.



EXERCISE PROBLEMS:

1) Express the following integrals in polar coordinates.

- A. $\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x + 2y + 3) dy dx$
- B. $\int_0^2 \int_{-\sqrt{9-x^2}}^0 (4 - x^2 - y^2) dy dx$
- C. $\int_{-4}^0 \int_0^{\sqrt{16-x^2}} (x + x^2 + y^2) dy dx$
- D. $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2 - yx^2 - y^3) dy dx$
- E. $\int_{-6}^0 \int_{-\sqrt{36-y^2}}^{\sqrt{36-y^2}} \left(\tan^{-1} \left(\frac{y}{x} \right) \right) dx dy$
- F. $\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \left(\frac{y}{x} \right) dy dx$

2) Express the following integrals in rectangular coordinates.

- A. $\int_0^{\pi} \int_0^2 (r^2)rdrd\theta$
 B. $\int_{\pi}^{2\pi} \int_0^3 (r\cos(\theta))rdrd\theta$
 C. $\int_0^{2\pi} \int_0^3 r^3drd\theta$
 D. $\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r^4\cos(\theta)drd\theta$
 E. $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^3 r^4\cos(\theta)drd\theta$
 F. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_0^3 r^2\sin(\theta)drd\theta$

3) The formula to represent a function f with inputs x and y and output z is $z = f(x, y) = x + y$ and we wish to obtain the volume below f and over the region $2 \leq r \leq 4, \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ in the xy plane.

- A. Convert the roof of our volume to polar coordinates and identify the volume we're seeking in your 3D kit.
 B. If there are two divisions in r and two divisions in θ and the volume is to be approximated using the minimum value for r and θ , find r_1, r_2, θ_1 and θ_2 and use them to fill in the following table with numerical values.

| Division | Length | Width | Height | Volume |
|----------|--------|-------|--------|--------|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

- C. Use the values of r_1, r_2, θ_1 and θ_2 to fill in the same table below using $r_1, r_2, \theta_1, \theta_2, \Delta r$, and $\Delta\theta$ instead of numerical values. (Note, the divisions should not change between the two tables.)

| Division | Length | Width | Height | Volume |
|----------|--------|-------|--------|--------|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

- D. Express the approximate volume numerically.
- E. Express the volume obtained in part D in the form $\sum \sum (...)\Delta r \Delta \theta$
- F. Take the appropriate limits to convert the sum in part D to an integral and evaluate the integral to obtain the precise volume.

4) The formula to represent the density of lead is given by $z = f(x, y) = x^2 + y^2 + x \frac{kg}{km^2}$ and we wish to obtain the amount of lead in the region $1 \leq r \leq 5, \frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ in the xy plane.

- A. Convert the roof of our volume to polar coordinates and identify the volume we're seeking in your 3D kit.
- B. If there are two divisions in r and two divisions in θ and the volume is to be approximated using the middle value for r and θ , find r_1, r_2, θ_1 and θ_2 and use them to fill in the following table with numerical values.

| Division | Length | Width | Height | Volume |
|----------|--------|-------|--------|--------|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

- C. Use the values of r_1, r_2, θ_1 and θ_2 to fill in the same table below using $r_1, r_2, \theta_1, \theta_2, \Delta r$, and $\Delta \theta$ instead of numerical values. (Note, the divisions should not change between the two tables.)

| Division | Length | Width | Height | Volume |
|----------|--------|-------|--------|--------|
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |

- D. Express the approximate mass of lead numerically.
- E. Express the mass of lead obtained in part D in the form $\sum \sum (...)\Delta r \Delta \theta$
- F. Take the appropriate limits to convert the sum in part E to an integral and evaluate the integral to obtain the precise mass of lead.